Quasidistributions in nonlinear quantum optics

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Abstract

We derive wave quasidistributions for compound optical parametric processes including nonclassical regimes and illustrate them using recent experimental data.

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1 Introduction

It is well known from the papers by R.J. Glauber [1, 2] published in 1963, founding the modern quantum optics, that optical processes having completely quantum behavior, i.e. having no classical analogue, are described by means of quasidistributions, e.g. using the weighting function in the diagonal Glauber-Sudarshan representation of the density matrix [3]. Such quasidistributions have some properties of classical distribution functions, e.g. they are normalized, some other properties are violated because they are in general generalized functions (linear functionals); they can be more singular than the Dirac function and they can take on negative values. Reviews of quasidistributions used in quantum optics can be found in books [4, 5]. This reflects the physical fact that the quantum dynamics are much more rich than the classical dynamics and hence for some quantum effects it may happen that there is a debt of probability expressed by negative values of the classical tool of probability function used for description of a quantum system. Such an approach represents important point of view based on wave properties of quantum systems.

We use optical parametric processes in nonclassical regimes to illustrate the tool of wave quasidistributions, where quantum-noise components take on negative values and when standard Cauchy integrals may fail. The so-called generalized superposition of coherent and quantum-noise fields can be adopted in this case [3].

2 Optical parametric processes

To illustrate our method, we use a general optical parametric process with strong classical coherent pumping in rotating-wave approximation involving optical parametric amplification, frequency conversion and second subharmonic generations described by the interaction Hamiltonian

$$\hat{H}_{\text{int}} = -\hbar g_1 \hat{a}_1 \hat{a}_2 - \hbar g_2 \hat{a}_1^\dagger \hat{a}_2 - \hbar g_3 \hat{a}_1^2 - \hbar g_4 \hat{a}_2^2 + \text{h.c.}, \tag{1}$$

where $g_j, j = 1, 2, 3, 4$ are the corresponding coupling constants proportional to the quadratic susceptibility $\chi^{(2)}$ and to the real amplitude of pumping, h.c. means the Hermitian conjugate terms; $\hat{a}_{1,2}$ ($\hat{a}_{1,2}^\dagger$) represent annihilation (creation) operators of a photon in the corresponding mode. The above mentioned processes are obtained as special cases.

Solving the Heisenberg equations of motion, we obtain their solution in the form [6]

$$\begin{aligned} \hat{a}_1(t) &= \hat{a}_1(0)u_1(t) + \hat{a}_1^\dagger(0)v_1(t) + \hat{a}_2(0)w_1(t) + \hat{a}_2^\dagger(0)y_1(t) + \hat{F}_1, \\ \hat{a}_2(t) &= \hat{a}_1(0)u_2(t) + \hat{a}_1^\dagger(0)v_2(t) + \hat{a}_2(0)w_2(t) + \hat{a}_2^\dagger(0)y_2(t) + \hat{F}_2, \end{aligned} \tag{2}$$

where the functions $u(t)$, $v(t)$, $w(t)$, $y(t)$ satisfy the corresponding identities following from the commutation rules for the interaction time t. Losses and noise can be described in the standard quantum consistent way assuming that both the modes are coupled to large Gaussian-Markovian reservoir systems (see, e.g., [3], chapter 7). In this case the field amplitudes are damped (losses) and additional contributions are obtained from reservoir Langevin forces (noise), which is described by the damping constants $\gamma_{1,2}$ and by the reservoir terms $\hat{F}_{1,2}$ arising from the Langevin forces and ensuring the commutation rules for the photon operators for all times including losses and noise.

Now we can use the normal quantum characteristic function

$$C_{\mathcal{N}}(\beta_1,\beta_2,t) = \mathrm{Tr}\{\hat{\rho}\exp(\beta_1\hat{a}_1^\dagger(t)+\beta_2\hat{a}_2^\dagger(t))\exp(-\beta_1^*\hat{a}_1(t)-\beta_2^*\hat{a}_2(t))\}, \quad (3)$$

which is obtained, using the above solutions, in the Gaussian form [3]

$$\begin{aligned} C_{\mathcal{N}}(\beta_1,\beta_2,t) &= \exp\Big[-|\beta_1|^2 B_1 - |\beta_2|^2 B_2 + D_{12}\beta_1^*\beta_2^* + D_{12}^*\beta_1\beta_2 \qquad (4)\\ &+ \bar{D}_{12}\beta_1\beta_2^* + \bar{D}_{12}^*\beta_1^*\beta_2 + \frac{1}{2}\left(C_1\beta_1^{*2}+C_1^*\beta_1^2+C_2\beta_2^{*2}+C_2^*\beta_2^2\right)\Big], \end{aligned}$$

where β_1, β_2 are parameters of the characteristic function; we can assume the spontaneous process without loss of generality. In this case the quantum noise functions in (4) are defined and obtained as

$$\begin{aligned} B_j &= \langle\Delta\hat{a}_j^\dagger\Delta\hat{a}_j\rangle = |v_j|^2+|y_j|^2+\langle\hat{F}_j^\dagger\hat{F}_j\rangle, j=1,2, \qquad (5)\\ C_j &= \langle(\Delta\hat{a}_j)^2\rangle) = u_jv_j+w_jy_j+\langle\hat{F}_j^2\rangle,\\ D_{12} &= \langle\Delta\hat{a}_1\Delta\hat{a}_2\rangle = u_1v_2+w_1y_2+\langle\hat{F}_1\hat{F}_2\rangle,\\ \bar{D}_{12} &= -\langle\Delta\hat{a}_1^\dagger\Delta\hat{a}_2\rangle = -v_1^*v_2-y_1^*y_2-\langle\hat{F}_1^\dagger\hat{F}_2\rangle. \end{aligned}$$

Normal characteristic function (4) in a single mode of photon pairs can be formulated in a matrix form with the help of the complex matrix

$$\hat{A} = \begin{pmatrix} -B_1 & C_1 & \bar{D}_{12}^* & D_{12}\\ C_1^* & -B_1 & D_{12}^* & \bar{D}_{12}\\ \bar{D}_{12} & D_{12} & -B_2 & C_2\\ D_{12}^* & \bar{D}_{12}^* & C_2^* & -B_2 \end{pmatrix}, \qquad (6)$$

a column vector $\hat{\beta} = (\beta_1,\beta_1^*,\beta_2,\beta_2^*)^T$ and Hermitian conjugated row vector $\hat{\beta}^\dagger$ as $C_{\mathcal{N}}(\hat{\beta}) = \exp(\hat{\beta}^\dagger\hat{A}\hat{\beta}/2)$. Its determinant K equals

$$\begin{aligned} K &= K_+K_-, \qquad (7)\\ K_+ &= B_{1+}B_{2+}-|D_{12+}|^2,\ K_- = B_{1-}B_{2-}-|D_{12-}|^2,\\ B_{1,2+} &= B_{1,2}+|C_{1,2}|,\ B_{1,2-} = B_{1,2}-|C_{1,2}|,\\ D_{12+} &= |D_{12}|-|\bar{D}_{12}|,\ D_{12-} = |D_{12}|+|\bar{D}_{12}|, \end{aligned}$$

provided that the following phase conditions are fulfilled

$$\arg(C_1) - \arg(D_{12}) + \arg(\bar{D}_{12}) = 0, \ \arg(C_2) - \arg(D_{12}) - \arg(\bar{D}_{12}) = 0. \quad (8)$$

If these phase differences are π, the signs in $D_{12\pm}$ at $|\bar{D}_{12}|$ are opposite. These particular phase conditions are well related to the impossibility to control phases in photocounting experiments and are automatically arranged during the interaction. We see that the increasing $|C_j|$ and $|\bar{D}_{12}|$ supports the positive value of K_+ and negative value of K_-, thus supporting the quantum effects. At the opposite phase they more or less compensate each other. In this case the nonlinear process is described by the transformed matrix

$$\hat{A} = \begin{pmatrix} -(B_1 + |C_1|) & 0 & 0 & |D_{12}| - |\bar{D}_{12}| \\ 0 & -(B_1 - |C_1|) & |D_{12}| + |\bar{D}_{12}| & 0 \\ 0 & |D_{12}| + |\bar{D}_{12}| & -(B_2 - |C_2|) & 0 \\ |D_{12}| - |\bar{D}_{12}| & 0 & 0 & -(B_2 + |C_2|) \end{pmatrix}, \quad (9)$$

having eigenvalues $\lambda_{1,2\pm} = -(B_{1\pm} + B_{2\pm})/2 \pm \left[(B_{1\pm} - B_{2\pm})^2 + 4D_{12\pm}^2\right]^{1/2}/2$ and for the same modes $\lambda_{1,2\pm} = -B_\pm \pm |D_{12\pm}|$, which is helpful for calculations of one-fold generating function and the corresponding distributions and moments. As special cases we have well-known results for particular processes when $C_1 = C_2 = \bar{D}_{12} = 0$ (parametric generation and amplification), $B_2 = C_2 = D_{12} = \bar{D}_{12} = 0$ (sub-harmonic generation), etc. In these processes it is always $B_{1,2} \geq 0$ as representing noise in single modes, whereas the quantum properties are characterized by $K = B_1 B_2 - |D_{12}|^2 < 0$ or $K = B_1^2 - |C_1|^2 < 0$, for classical cases $K \geq 0$. In our more general case mixing the basic processes, also $B_{1,2-}$ can be negative, which provides more general situations.

3 Generating function

The s-ordered generating function is obtained from the s-ordered characteristic function as [3]

$$G_s(\lambda_1, \lambda_2, t) = \frac{1}{\pi^2 \lambda_1 \lambda_2} \int\int \exp\left(-\frac{|\beta_1|^2}{\lambda_1} - \frac{|\beta_2|^2}{\lambda_2}\right) C_s(\beta_1, \beta_2, t) \mathrm{d}^2\beta_1 \mathrm{d}^2\beta_2, \tag{10}$$

where λ_1 and λ_2 are parameters of the generating function and the s-ordered characteristic function is defined by substituting $B_{1,2s\pm} = B_{1,2\pm} + (1-s)/2$ instead of $B_{12\pm}$. For M equally behaved modes (temporal, spatial and polarization in the spirit of Mandel-Rice formula) we obtain the s-ordered generating function in the form [6]

$$\begin{aligned} G_s(\lambda_1, \lambda_2, t) &= G_{s+}(\lambda_1, \lambda_2, t) G_{s-}(\lambda_1, \lambda_2, t) \qquad (11) \\ &= (1 + \lambda_1 B_{1s+} + \lambda_2 B_{2s+} + \lambda_1 \lambda_2 K_{s+})^{-M/2} \\ &\quad \times (1 + \lambda_1 B_{1s-} + \lambda_2 B_{2s-} + \lambda_1 \lambda_2 K_{s-})^{-M/2}, \end{aligned}$$

taking into account the above phase conditions enabling us to write the above characteristic function as a product of two characteristic functions for down-conversion processes, each for half number of modes, $M/2$. For normal ordering $s = 1$. Possible negative values of $B_{1,2s-}$ and $K_{s\pm}$ play no role in calculating the generating function because the corresponding Gaussian integrals can always be calculated for sufficiently small $\lambda_{1,2}$ and the analytic continuation can be used to define the generating function for all $\lambda_{1,2}$ [7–9] (the generating function is analytic as a consequence of nonnegative values of the optical intensity).

4 Joint photon-number distribution

Joint photon-number distribution $p(n_1, n_2, t)$ for multi-Gaussian field with M degrees of freedom can be derived in the form [6, 10, 11]:

$$p(n_1, n_2, t) = \sum_{k=0}^{n_1} \sum_{l=0}^{n_2} p_+(k, l, t) p_-(n_1 - k, n_2 - l, t), \tag{12}$$

where

$$\begin{aligned} p_\pm(n_1, n_2, t) &= \frac{(-1)^{n_1+n_2}}{n_1! n_2!} \left. \frac{\partial^{n_1+n_2} G_{\mathcal{N}\pm}(\lambda_1, \lambda_2, t)}{\partial \lambda_1^{n_1} \partial \lambda_2^{n_2}} \right|_{\lambda_1=\lambda_2=1} \\ &= \frac{1}{\Gamma(M/2)} \frac{(B_{1\pm} + K_\pm)^{n_1} (B_{2\pm} + K_\pm)^{n_2}}{(1 + B_{1\pm} + B_{2\pm} + K_\pm)^{n_1+n_2+M/2}} \\ &\times \sum_{r=0}^{\min(n_1,n_2)} \frac{\Gamma(n_1 + n_2 + M/2 - r)}{r!(n_1 - r)!(n_2 - r)!} \\ &\times \frac{(-K_\pm)^r [(1 + B_{1\pm} + B_{2\pm} + K_\pm)]^r}{[(B_{1\pm} + K_\pm)(B_{2\pm} + K_\pm)]^r}. \end{aligned} \tag{13}$$

Determinants $K_\pm = B_{1\pm} B_{2\pm} - |D_{12\pm}|^2$ are crucial for the judgement of classicality or nonclassicality of the field, as discussed above. Negative values of the determinants $K_\pm$ mean that a given field cannot be described classically, which is the case of the field under discussion composed of photon pairs. In (13), the quantities $B_{1,2\pm} + K_\pm$ can be considered as characteristics of fictitious noise present in the fields, giving declination from the purity of the process, i.e. from the diagonal form of $p(n_1, n_2, t)$ when photons are ideally paired; physically they are positive. However, in this case they characterize virtual distributions giving the actual distribution after the convolution process, which means that they may take on also negative values in some cases. Even if the corresponding partial joint photon-number distribution $p_-(n_1, n_2, t)$ can be unphysical taking on negative values, the resulting convoluted distribution is actual quantum oscillating physical distribution (this is an analogue of one-dimensional case [3]), i.e. this leads to quantum oscillations in the photon-number distribution $p(n_1, n_2, t)$ [6] (see Figs. 2–4 in Sec. 8). The expressions (13) are valid regardless of the sign of the quantities B and K. This situation is qualitatively changed in the wave joint distributions involving Fourier transformations.

We can mention that we can describe Raman scattering with strong coherent pumping when eliminating phonon variables in the same way [3].

5 Joint wave quasidistributions

The s-ordered joint wave quasidistribution $P_s(W_1, W_2, t)$ is obtained from the partial quasidistributions in the form of convolution

$$P_s(W_1, W_2, t) = \int_0^{W_1} \mathrm{d}W_1' \int_0^{W_2} \mathrm{d}W_2' P_{s+}(W_1', W_2', t) P_{s-}(W_1 - W_1', W_2 - W_2', t). \quad (14)$$

Provided that the s-ordered determinant $K_{s\pm} = B_{1s\pm}B_{2s\pm} - |D_{12\pm}|^2 = K_\pm + (1-s)(B_{1s\pm} + B_{2s\pm})/2 + (1-s)^2/4$ is positive the s-ordered joint quasidistribution $P_{s\pm}(W_1, W_2)$ of integrated intensities exists as an ordinary function [10] which cannot take on negative values:

$$\begin{aligned} P_{s\pm}(W_1, W_2, t) &= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp(-\mathrm{i}s_1 W_1 - \mathrm{i}s_2 W_2) G_{s\pm}(-\mathrm{i}s_1, -\mathrm{i}s_2, t) \mathrm{d}s_1 \mathrm{d}s_2 \\ &= \frac{1}{\Gamma(M/2) K_{s\pm}^{M/2}} \left(\frac{K_{s\pm}^2 W_1 W_2}{|D_{12\pm}|^2} \right)^{(M/2-1)/2} \\ &\quad \times \exp\left[-\frac{(B_{2s\pm} W_1 / B_{1s\pm} + W_2) B_{1s\pm}}{K_{s\pm}} \right] \\ &\quad \times I_{M/2-1}\left(2\sqrt{\frac{|D_{12\pm}|^2 W_1 W_2}{K_{s\pm}^2}} \right). \end{aligned} \quad (15)$$

If the s-ordered determinant $K_{s\pm}$ is negative, the joint signal-idler quasidistribution $P_{s\pm}$ of integrated intensities exists in general as a generalized function that can take on negative values or can even have singularities. It can be obtained in the following form [10]:

$$
\begin{aligned}
P_{s\pm}(W_1, W_2, t) \;=\; & \frac{(W_1 W_2)^{(M/2-1)/2}}{\pi\Gamma(M/2)(-K_{s\pm})^{1/2}(B_{1s\pm}B_{2s\pm})^{M/4}} \\
& \times \exp\left(-\frac{W_1}{2B_{1s\pm}} - \frac{W_2}{2B_{2s\pm}}\right) \mathrm{sinc}\left[A\left(\frac{B_{2s\pm}}{B_{1s\pm}}W_1 - W_2\right)\right];
\end{aligned}
\tag{16}
$$

$\mathrm{sinc}(x) = \sin(x)/x$, $A = (-K_{s\pm}B_{2s\pm}/B_{1s\pm})^{-1/2}$. Oscillating behavior is typical for the quasidistribution $P_{s\pm}$ written in (16). Compared to our considerations in [10] the formula (16) is exact because we can show that the poles in the upper half complex plane give generally no contribution to the corresponding integral and one also need not assume that $-K_{s\pm}$ is much less than $B_{1,2s\pm}$ in the expressions (see next section).

There exist threshold values $s_{\mathrm{th}\pm}$ of the ordering parameter s for given values of parameters $B_{1,2\pm}$ and $D_{12\pm}$ determined by $K_{s\pm} = 0$ (the corresponding joint s-ordered wave distribution is diagonal):

$$
s_{\mathrm{th}\pm} = 1 + B_{1\pm} + B_{2\pm} - \sqrt{(B_{1\pm} + B_{2\pm})^2 - 4K_{\pm}}; \tag{17}
$$

$-1 \leq s_{\mathrm{th}\pm} \leq 1$. Quasidistributions $P_{s\pm}$ for $s \leq s_{\mathrm{th}\pm}$ are ordinary functions with non-negative values whereas those for $s > s_{\mathrm{th}\pm}$ are generalized functions with negative values and oscillations. This expression can also be written in the form $1 + B_{1\pm} + B_{2\pm} - \sqrt{(B_{1\pm} - B_{2\pm})^2 + 4D_{12\pm}^2}$ directly related to the principal squeeze parameter for equally behaved modes $\lambda = 1 + 2(B_{\pm} - |D_{12\pm}|)$. In this sense the quasidistribution behaves in a quantum way for s between the threshold value s_{th} and 1 (Glauber-Sudarshan quasidistribution), for s equal or less than the threshold value, the vacuum fluctuations related to the value s compensate nonclassicality in the field (expressed e.g. by the squeezing of vacuum fluctuations in the field) and the quasidistribution behaves classically.

Considering the basic process only, even if negative probabilities, which can be reconstructed from experimental data ([17] and references therein), represent only qualitative phenomenon reflecting debt of probabilities in richer quantum dynamics compared to classical dynamics for us and we do not interpret them directly, they have direct consequences in the discrete region in $p(n_1, n_2)$: In the quantum region $K < 0$ and $K + B \geq 0$, i.e. $K + B < B$; in the pure case we have $K = -B$ and $K + B = 0$, which leads to the diagonal Mandel-Rice formula for $p(n_1, n_2)$ giving the oscillating sum-number distribution with zero odd values; in general we have quantum oscillations and squeezed-form distribution up to the border between quantum and classical regions for $K = 0$ ($K + B = B$) which reflect negative probabilities. Then in the classical region $K \geq 0$, i.e. $K + B \geq B$ quasidistributions behave classically and forms of $p(n_1, n_2)$ change among that for $K = 0$ and the isotropic case $p(n_1, n_2) = p(n_1)p(n_2)$.

We can mention that quantum entanglement equivalent to the above quantum phenomena is obtained if $K_+K_{\mathcal{A}+} + K_-K_{\mathcal{A}-} < 0$, where $K_{\mathcal{A}\pm}$ are the corresponding determinants for antinormal operator ordering, i.e. $B_\pm$ is substituted by $B_\pm + 1$ [6]. A number of other quantities can be derived from the basic joint distributions [6], such as conditional number distributions $p_{c,2}(n_2; n_1)$ which are sub-Poissonian giving the Fano factor $F_{c,2} < 1$, sub-Poissonian difference-number distribution $p_-(n)$, sum- and difference-wave quasidistributions $P_{s\pm}(W)$ exhibiting classical and quantum behavior, respectively; the principal squeezing $\lambda = 1 + B_1 + B_2 - 2\mathrm{Re}\bar{D}_{12} - |C_1 + C_2 + 2D_{12}| < 1$ (for the same modes $\lambda \geq s_{\mathrm{th}-}$) and sub-shot-noise behavior $R = 1 + (K_+ + K_-)/2B < 1$ (the same modes) are also obtained.

6 Forms of quasidistributions

When we simplify our denotation of the quantities $B_{1,2s\pm}$, $D_{12\pm}$, $K_{s\pm}$ to $B_{1,2s}$, $D_{1,2}$, K_s, the above solutions are two of eight possibilities to have $B_{1,2s}$ and K_s positive or negative:

1. $B_{1,2s} > 0$, $K_s > 0$

In this case the integrals

$$P_s(W_1, W_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\exp(-\mathrm{i}s_1 W_1 - \mathrm{i}s_2 W_2)}{[1 - \mathrm{i}s_1 B_{1s} - \mathrm{i}s_2 B_{2s} - s_1 s_2 K_s]^M} \mathrm{d}s_1 \mathrm{d}s_2 \tag{18}$$

with respect to variables s_1 and s_1 are in the lower complex half-plane $\Pi^{(-)}$ and applying the Cauchy integral two times, we obtain the above I_M-distribution (15) which is regular and non-negative. We see this writing the polynomial in the denominator as

$$\begin{aligned} 1 - \mathrm{i}s_1 B_{1s} - \mathrm{i}s_2 B_{2s} - s_1 s_2 K_s &= -(\mathrm{i}B_{1s} + s_2 K_s) \\ &\quad \times \left[s_1 + \frac{s_2 |D_{12}|^2 + \mathrm{i}(B_{1s} + s_2^2 B_{2s} K_s)}{B_{1s}^2 + s_2^2 K_s^2} \right]. \end{aligned} \tag{19}$$

Under these conditions the system behaves classically.

2. $B_{1,2s} > 0$, $K_s < 0$

This is the standard quantum case. To have the pole in $\Pi^{(-)}$ in the integration along s_1 to use the Cauchy integral, it must be

$$B_{1s} - s_2^2 B_{2s}(-K_s) > 0, \tag{20}$$

i.e. s_2 is filtered out the interval $(-A, +A)$, $A = (B_{1s}/B_{2s}(-K_s))^{1/2}$ and we arrive at the integral

$$P_s(W_1, W_2) = \frac{W_1^{M-1}}{2\pi\Gamma(M)} \int_{-A}^{+A} \frac{\exp[-\mathrm{i}s_2 W_2 - \frac{1 - \mathrm{i}s_2 B_{2s}}{-\mathrm{i}s_2 K_s + B_{1s}} W_1]}{(-\mathrm{i}s_2 K_s + B_{1s})^M} \mathrm{d}s_2. \tag{21}$$

Now we can show that the pole $s_2 = -\mathrm{i}B_{1s}/K_s$ lying in the upper half-plane $\Pi^{(+)}$ in this case gives no contribution to the integral (21) which is analytic in the lower half-plane $\Pi^{(-)}$. If we integrate the integral (21) with respect to W_2 (generalized function) and use one factor from the denominator, we have

$$\frac{1}{-\mathrm{i}s_2(-\mathrm{i}s_2K_s + B_{1s})} = \left(\frac{1}{-\mathrm{i}s_2} - \frac{K_s}{-\mathrm{i}s_2K_s + B_{1s}}\right)\frac{1}{B_{1s}}. \tag{22}$$

The integral involving the second term is zero because taking it from $-\infty$ to $+\infty$ (the integral for the filtered values of s_2 is zero because this means the opposite inequality in (20) and the pole lies in the upper half-plane), the pole is at $s_2 = -\mathrm{i}B_{1s}/K_s$ in $\Pi^{(+)}$ and performing the derivative with respect to W_2 we have the same integral with the factor in the denominator decreased by one; hence one factor $-\mathrm{i}s_2K_s + B_{1s}$ in the denominator is replaced by B_{1s}. Successively we replace all these factors by B_{1s} including the corresponding terms in the exponential function decomposing it into the series. We finally obtain exactly the above sinc-quasidistribution (16) after explicit symmetrization (we change the order of integrations along s_1 and s_2 obtaining the same result, multiply the results and take the square root) , i.e.

$$\begin{aligned} P_s(W_1, W_2) &= \frac{(W_1W_2)^{(M-1)/2}}{\pi\Gamma(M)(-K_s)^{1/2}(B_{1s}B_{2s})^{M/2}} \exp\left(-\frac{W_1}{2B_{1s}} - \frac{W_2}{2B_{2s}}\right) \\ &\quad \times \mathrm{sinc}\left[A(\frac{B_{2s}}{B_{1s}}W_1 - W_2)\right]. \end{aligned} \tag{23}$$

We note that we have started by the integration along s_1. If we start with s_2, everything is valid changing mutually indices 1 and 2.

Respecting the simplified denotation $B_{1,2s}$ for $B_{1,2s-} = B_{1,2s} - |C_{1,2}|$ which can also be negative in principle, e.g. in single-mode sub-harmonic generation in nonclassical regime, where $B^2 - |C|^2 < 0$, we can have further six cases for our discussion of quasidistributions in the general optical parametric process. Of course, in the following conditions K_s means K_{s-}.

For $B_{1,2s+} > 0$ only cases 1,2 are appropriate and the resulting wave distributions are convolutions of the partial distributions.

3. $B_{1s} < 0,\ B_{2s} > 0,\ K_s > 0$

To have the pole in s_1 in $\Pi^{(-)}$ we must have similarly to (20)

$$-(-B_{1s}) + s_2^2 B_{2s} K_s > 0, \tag{24}$$

which means that $|s_2| > -A_1$, $A_1 = (-B_{1s}/B_{2s}K_s)^{1/2}$, s_2 is not restricted and takes on all values on the real axis $(-\infty, +\infty)$. The integral in (18) with respect to s_1 is the Cauchy integral, however the pole in s_2 lies in $\Pi^{(+)}$ and the integral in s_2 is zero. The quasidistribution $P_s(W_1, W_2)$ has a point support and cannot directly be determined by the Cauchy theorem. However, when changing the order of integrations integrating first along s_2 and then along s_1, cases 3 and 4 are interchanged and both the quasidistributions are determined in case 4.

4. $B_{1s} > 0, B_{2s} < 0, K_s > 0$

In this case we have

$$B_{1s} - s_2^2(-B_{2s})K_s > 0, \tag{25}$$

$|s_2| < A_2$, $A_2 = (B_{1s}/(-B_{2s})K_s)^{1/2}$, the pole in s_2 is in $\Pi^{(-)}$, and consequently $P_s(W_1, W_2)$ is expressed as the above finite integral (21) in s_2, however the pole in s_2 cannot be neglected as in case 2.

5. $B_{1s} < 0,\ B_{2s} < 0,\ K_s > 0$

In this case

$$-(-B_{1s}) - s_2^2(-B_{2s})K_s > 0, \tag{26}$$

which means that $-s_2^2 > A_3^2$, $A_3 = (-B_{1s}/(-B_{2s})K_s)^{1/2}$, which cannot be fulfilled on the real axis s_2; the quasidistribution could be found in the general complex plane $s_2{}' = -\mathrm{i}s_2$. The pole in s_2 lies in $\Pi^{(+)}$.

6. $B_{1s} < 0$, $B_{2s} > 0$, $K_s < 0$

We have

$$-(-B_{1s}) - s_2^2 B_{2s}(-K_s) > 0, \tag{27}$$

giving $-s_2^2 > A_4^2$, $A_4 = (-B_{1s}/B_{2s}(-K_s))^{1/2}$, and the condition cannot again be fulfilled on the real axis s_2 as in case 5; the pole in s_2 is in $\Pi^{(-)}$.

7. $B_{1s} > 0$, $B_{2s} < 0$, $K_s < 0$

We have

$$B_{1s} + s_2^2(-B_{2s})(-K_s) > 0, \tag{28}$$

and $-s_2^2 < A_5^2$, $A_5 = (B_{1s}/(-B_2 s)(-K_s))^{1/2}$ is fulfilled for all real s_2; the pole in s_2 lies in $\Pi^{(+)}$. Thus s_2 is not filtered and the Cauchy integral is zero as in case 3. However, from the symmetry changing the order of integrations, we have case 6.

8. $B_{1s} < 0$, $B_{2s} < 0$, $K_s < 0$

This is physically more interesting case compared to cases 5–7 above. In this case the necessary condition is

$$-(-B_{1s}) + s_2^2(-B_{2s})(-K_s) > 0, \tag{29}$$

with the pole in s_2 in $\Pi^{(-)}$. Thus $|s_2| > -A_6$, $A_6 = (-B_{1s}/(-B_2 s)(-K_s))^{1/2}$. The integration in s_2 is not restricted as in case 3 and $P_s(W_1, W_2)$ is of the form of I_{M-1}-distribution (15), however with respect to $K_s < 0$, it changes the sign in dependence on the number of modes M, thus being the quasidistribution taking on negative values in these cases.

The above arguments are also correct for the corresponding generating function (4) in [11] and the joint distribution (13) appropriate for the optical parametric process stimulated by a coherent light. The most general description can combine the results from [11] and [6] giving wave quasidistributions for a general optical parametric process stimulated by chaotic and squeezed light with initial coherent stimulating components.

7 One-dimensional process

As an illustration we can now give the simplest case of the single mode second subharmonic generation ($\lambda_1 = \lambda$, $\lambda_2 = 0$, $B_1 = B$, $C_1 = C$, $B_2 = C_2 = D_{12} = \bar{D}_{12} = 0$).

i) Spontaneous process

In this case the normal generating function has the form [3]

$$G_{\mathcal{N}}(-\mathrm{i}s) = \frac{1}{[1 - \mathrm{i}s(B + |C|)]^{M/2}[1 - \mathrm{i}s(B - |C|)]^{M/2}} \tag{30}$$

for the degenerate case (for M even when the cross-correlation coefficient D_{12} stands instead of C we also have the generating function for non-degenerate case). Clearly the second-order polynomial in the denominator can be written as $1-2\mathrm{i}Bs-Ks^2$ (s is now Fourier variable and not s-ordering parameter; generalization to the s-ordering is straightforward writing $B_s = B+(1-s)/2$ instead of B), where the determinant $K = B^2 - |C|^2$ (in non-degenerate case $K = B_1B_2 - |D_{12}|^2$). If the field behaves nonclassically ($K < 0$), the mean number of quantum-noise photons $F = B + |C|$ is positive and the corresponding distribution is the Rayleigh gamma distribution [3], whereas $E = B - |C|$ is negative, leading to sub-Poisson behavior, squeezing of vacuum fluctuations and quantum oscillations in photon-number distribution. Attempt to solve the problem of determining the quasidistribution in nonclassical region in one dimension was not successful [13]. Generalizing the polynomial to the two dimensional form as $1 - \mathrm{i}Bs_1 - \mathrm{i}Bs_2 - s_1 s_2 K$ and performing the inverse Fourier transforms after s_1 and s_2 determining the Glauber-Sudarshan wave distribution $P_{\mathcal{N}}(W_1, W_2)$, we obtain again filtering of values of s_2 for $K < 0$ to have the Cauchy integral after s_1 with the pole in $\Pi^{(-)}$ as above, giving the frequency filtering $|s_2| \leq (-K)^{-1/2}$. Thus we have the partial integral

$$P_{\mathcal{N}-}(W) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp(-\mathrm{i}sW)}{(1 - \mathrm{i}sE)^{M/2}} \mathrm{d}s. \tag{31}$$

Using the well-know indentity ($P\frac{1}{s}$ means the principal value of the integral) $\int_0^\infty \exp(-\mathrm{i}sW)\mathrm{d}W = -\mathrm{i}P\frac{1}{s} + \pi\delta(s)$, we verify that $\int_0^\infty P_{\mathcal{N}-}(W)\mathrm{d}W = 1$. Integrating and deriving this inegral with respect to W as above and using the

$$\frac{1}{-\mathrm{i}s\,(1 - \mathrm{i}sE)} = \frac{1}{-\mathrm{i}s} - \frac{E}{1 - \mathrm{i}sE}, \tag{32}$$

we obtain successively after integration in $\Pi^{(-)}$ and taking only not filtered values of s:

$$P_{\mathcal{N}-}(W) = \frac{\sin(AW)}{\pi W},\ A = (-K)^{-1/2}. \tag{33}$$

The resulting quasidistribution is the convolution of the Rayleigh gamma distribution related to F [3],

$$P_{\mathcal{N}+}(W) = \frac{W^{M/2-1}}{\Gamma(M/2)F^{M/2}} \exp\left(-\frac{W}{F}\right), \tag{34}$$

and of this sinc-quasidistribution taking on negative values. For non-degenerate process $E = B_2 - |D_{12}|$, $F = B_1 + |D_{12}|$, $A = (-B_1/B_2K)^{1/2}$.

ii) Stimulated process

In this case [3]

$$G_{\mathcal{N}-}(-\mathrm{i}s) = \frac{1}{(1 - \mathrm{i}sE)^{M/2}} \exp\left(\frac{\mathrm{i}s\mathcal{A}}{1 - \mathrm{i}sE}\right), \tag{35}$$

where E is the above quantum noise component and $\mathcal{A}$ is a coherent component related to the initial field [3]. Decomposing the exponential function and applying the above arguments, we arrive at the shifted sinc-quasidistribution

$$P_{\mathcal{N}-}(W) = \frac{\sin(A(W-\mathcal{A}))}{\pi(W-\mathcal{A})}. \quad (36)$$

The resulting quasidistribution is the convolution of the regular I_{M-1}-distribution related to F [3],

$$P_{\mathcal{N}+}(W) = \frac{1}{F}\left(\frac{W}{\mathcal{B}}\right)^{(M/2-1)/2} \exp\left(-\frac{W+\mathcal{B}}{F}\right) I_{M/2-1}\left(2\frac{\sqrt{W\mathcal{B}}}{F}\right), \quad (37)$$

and of this sinc-quasidistribution taking on negative values; here $\mathcal{B}$ is another partial coherent component related to the initial field [3].

8 Illustrations

We have adopted experimental data from [10] obtained in the Joint Laboratory of Optics in Olomouc giving $B_1 = B_2 = 0.501$, $|D_{12}| = 0.8$, $M = 29$, adding $|C_1| = |C_2| = |\bar{D}_{12}| = 0.03$; then $B_{1+} = B_{2+} = 0.531$, $B_{1-} = B_{2-} = 0.471$, $D_{12+} = 0.77$, $D_{12-} = 0.83$. This leads to values $K = -0.389$, $s_{\mathrm{th}+} = 0.522$, $s_{\mathrm{th}-} = 0.282$, $K_+ = -0.311$, $K_- = -0.467$, for $s = 0.6$ $K_{s+} = -0.0585$, $K_{s-} = -0.238$, thus $K_\pm + B_\pm > 0$. Both partial wave quasidistributions $P_{s\pm}(W_1, W_2)$ exhibit nonclassical negative values and the resulting quasidistribution $P_s(W_1, W_2)$ is shown in Fig. 1. For s less than the threshold values we obtain the convolution of the $I_{M/2-1}$-regular and positive distributions behaving classically without oscillations.

Solving the system of equations of motion [6], we arrive for the "optimalized" value $g_1 t = 0.25$ choosing $g_3/g_1 = g_4/g_1 = g_2/g_1 = 0.1$, relative damping constants $\gamma_1/g_1 = \gamma_2/g_1 = 0.1$, and external reservoir noise

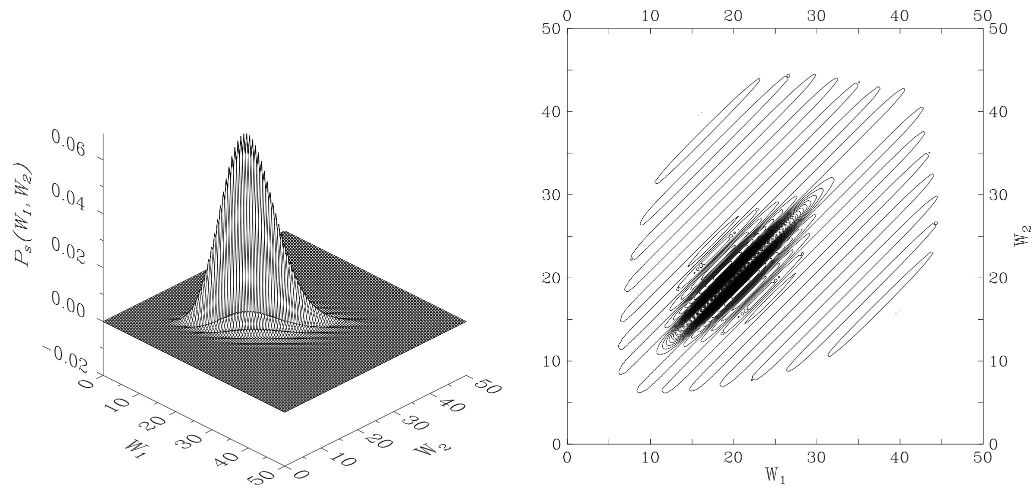

Figure 1: Joint resulting wave quasidistribution $P_s(W_1, W_2)$ (left) and its contour curves (right) for the values of given in the text (after [6]).

$\langle n_{\mathrm{d}1}\rangle = \langle n_{\mathrm{d}2}\rangle = 1$ to the following values of the parameters: $B_1 = B_2 = 0.0885$, $|C_1| = |C_2| = 0.0578$, $|D_{12}| = 0.258$, $|\bar{D}_{12}| = 0.0239$; C_j, D_{12} were nearly purely imaginary whereas $\bar{D}_{12}$ was practically real, so the above phase conditions are really fulfilled; we use $M = 10$. This provides the following values of the derived quantities: $B_{1,2+} = 0.146$, $B_{1,2-} = 0.0307$, $D_{12+} = 0.234$, $D_{12-} = 0.282$, $K_+ = -0.0332$, $K_- = -0.0784$, choosing $s = 0.6$ ($s_{\mathrm{th}+} = 0.825$, $s_{\mathrm{th}-} = 0.498$), $K_{s+} = 0.0168$, $K_{s-} = -0.261$. Now we see that $K_+ + B_+ > 0$, but $K_- + B_- < 0$, which virtually can occur leading to strong quantum oscillation to negative values of the distribution $p_-(n_1, n_2)$ and to the resulting physical distribution $p(n_1, n_2)$ as the convolution of p_+ and p_-. Fig. 2 provides quantum oscillating actual joint photon-number distribution obtained as the convolution of the partial virtual distributions p_+ and p_- given in Figs. 3 and 4, respectively; in Fig. 4 high negative oscillations of p_- are exhibited caused by negative values of $K_- + B_-$. This is

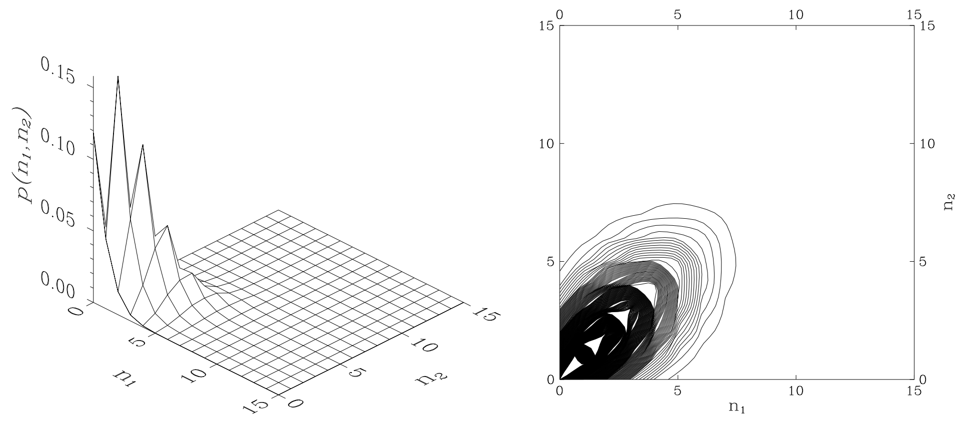

Figure 2: Time evolution of joint photon-number distribution $p(n_1, n_2)$ (left) for input parameters given in the text and its contour curves (right) (after [6]).

analogous to the one-dimensional case [12-14]. One can follow time evolution of this nonclassical distributions observing first a flat distribution at the beginning of the interaction developing as a regular distribution with $K_s > 0$ with progressing squeezing of the form to one line with $K_s = 0$ (classical diagonal distribution) and then successively demonstrating quantum oscillations when $K_s < 0$ approaching finally some steady-state squeezed-form of quasidistribution with increasing period of oscillations.

Except the above mentioned experimental data this approach has been applied to the interpretation of experimental data from various measurements, e.g. [15-17].

9 Conclusion

We have demonstrated nonclassical behavior of wave quasidistributions for general optical parametric processes as well as for some simpler particular

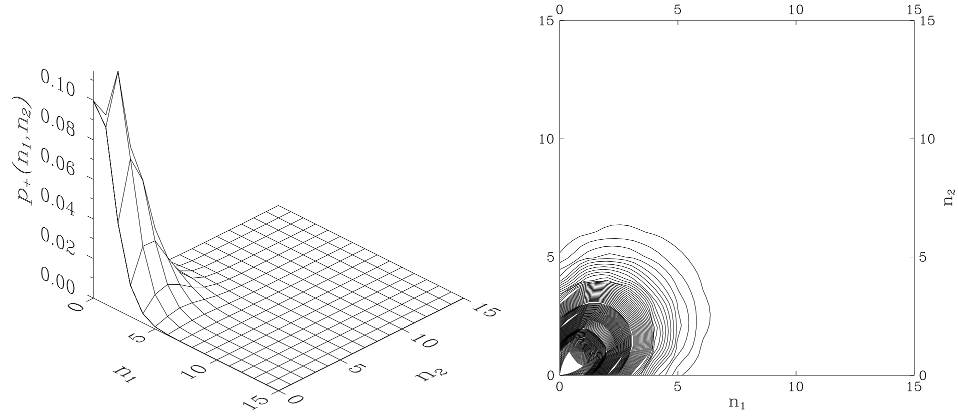

Figure 3: Time evolution of partial joint photon-number distribution $p_+(n_1, n_2)$ (left) for quantities as in Fig. 2 and its contour curves (right) (after [6]).

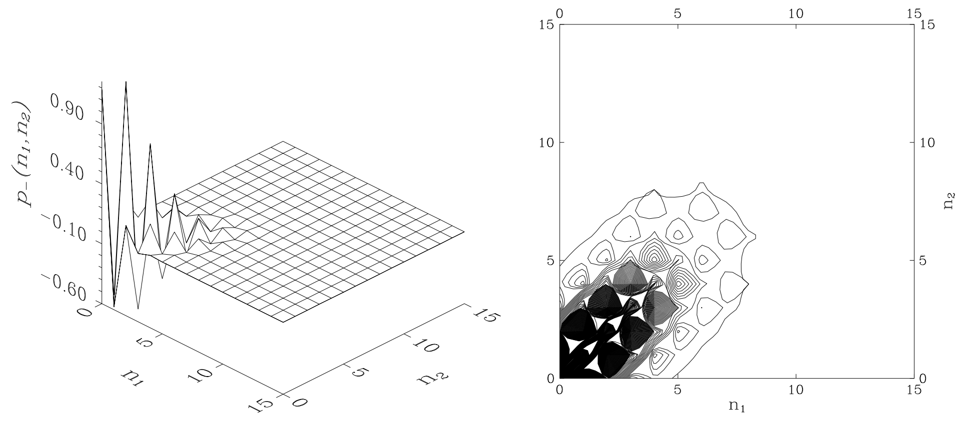

Figure 4: Time evolution of partial joint photon-number distribution $p_-(n_1, n_2)$ (left) for quantities as in Fig. 2 and its cut curves (right) (after [6]).

processes. We have discussed various regimes of the process in relation to various levels of noise in classical and nonclassical regimes and illustrated the system behavior by means of joint photon-number and wave distributions, also for some experimental data. We have demonstrated the important role of the sinc-quasidistributions in this description.

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