Quasidistributions in nonlinear quantum optics

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Introduction

- Papers by R.J. Glauber 1963.
- Modern quantum optics.
- Optical processes having quantum behavior, i.e. having no classical analogue.
- Quasidistributions, e.g. diagonal Glauber-Sudarshan representation of the density matrix.
- They are normalized, some other properties are violated because they are in general generalized functions (linear functionals); they can be more singular than the Dirac function and they can take on negative values.
- The reflection of the physical fact that the quantum dynamics are much more rich than the classical dynamics and hence for some quantum effects it may happen that there is a debt of probability expressed by negative values of the classical tool of probability function used for description of a quantum system. Such an approach represents important point of view based on wave properties of quantum systems.

- We use optical parametric processes in nonclassical regimes to illustrate the tool of wave quasidistributions, where quantum-noise components take on negative values and when standard Cauchy integrals may fail. The so-called generalized superposition of coherent and quantum-noise fields can be adopted in this case.
- Raman scattering can be described in a similar way.

Optical parametric processes

Interaction Hamiltonian

$$\hat{H}_{\rm int} = -\hbar g_1 \hat{a}_1 \hat{a}_2 - \hbar g_2 \hat{a}_1^\dagger \hat{a}_2 - \hbar g_3 \hat{a}_1^2 - \hbar g_4 \hat{a}_2^2 + {\rm h.c.} \quad (1)$$

Solutions of the Heisenberg equations

$$\hat{a}_1(t) = \hat{a}_1(0) u_1(t) + \hat{a}_1^\dagger(0) v_1(t) + \hat{a}_2(0) w_1(t) + \hat{a}_2^\dagger(0) y_1(t) + \hat{F}_1 \quad (2)$$
$$\hat{a}_2(t) = \hat{a}_1(0) u_2(t) + \hat{a}_1^\dagger(0) v_2(t) + \hat{a}_2(0) w_2(t) + \hat{a}_2^\dagger(0) y_2(t) + \hat{F}_2$$

The normal quantum characteristic function

$$C_{\mathcal{N}}(\beta_1, \beta_2, t) = {\rm Tr}\{\hat{\rho} \exp(\beta_1 \hat{a}_1^\dagger(t) + \beta_2 \hat{a}_2^\dagger(t)) \exp(-\beta_1^* \hat{a}_1(t) - \beta_2^* \hat{a}_2(t))\} \quad (3)$$

Its Gaussian form

$$C_{\mathcal{N}}(\beta_1,\beta_2,t) = \exp\Big[-|\beta_1|^2 B_1 - |\beta_2|^2 B_2 + D_{12}\beta_1^*\beta_2^* + D_{12}^*\beta_1\beta_2 + \bar{D}_{12}\beta_1\beta_2^* + \bar{D}_{12}^*\beta_1^*\beta_2 + \frac{1}{2}\left(C_1\beta_1^{*2} + C_1^*\beta_1^2 + C_2\beta_2^{*2} + C_2^*\beta_2^2\right)\Big], \quad (4)$$

where β_1, β_2 are parameters of the characteristic function; quantum noise functions

$$\begin{aligned} B_j &= \langle \Delta\hat{a}_j^\dagger \Delta\hat{a}_j\rangle = |v_j|^2 + |y_j|^2 + \langle \hat{F}_j^\dagger \hat{F}_j\rangle, j = 1,2, \\ C_j &= \langle (\Delta\hat{a}_j)^2\rangle) = u_j v_j + w_j y_j + \langle \hat{F}_j^2\rangle, \\ D_{12} &= \langle \Delta\hat{a}_1 \Delta\hat{a}_2\rangle = u_1 v_2 + w_1 y_2 + \langle \hat{F}_1 \hat{F}_2\rangle, \\ \bar{D}_{12} &= -\langle \Delta\hat{a}_1^\dagger \Delta\hat{a}_2\rangle = -v_1^* v_2 - y_1^* y_2 - \langle \hat{F}_1^\dagger \hat{F}_2\rangle \end{aligned} \quad (5)$$

Alternatively we can use the Schrödinger picture obtaining directly equations for $B_j, C_j, D_{12}, \bar{D}_{12}$ from the generalized Fokker-Planck equation.

Matrix formulation

$$\hat{A} = \begin{pmatrix} -B_1 & C_1 & \bar{D}_{12}^* & D_{12} \\ C_1^* & -B_1 & D_{12}^* & \bar{D}_{12} \\ \bar{D}_{12} & D_{12} & -B_2 & C_2 \\ D_{12}^* & \bar{D}_{12}^* & C_2^* & -B_2 \end{pmatrix}, \qquad (6)$$

a column vector $\hat{\beta} = (\beta_1, \beta_1^*, \beta_2, \beta_2^*)^T$ and Hermitian conjugated row vector $\hat{\beta}^\dagger$, then $C_{\mathcal{N}}(\hat{\beta}) = \exp(\hat{\beta}^\dagger \hat{A} \hat{\beta}/2)$. Its determinant K equals

$$\begin{aligned} K &= K_+ K_-, \qquad (7) \\ K_+ &= B_{1+}B_{2+} - |D_{12+}|^2,\ K_- = B_{1-}B_{2-} - |D_{12-}|^2, \\ B_{1,2+} &= B_{1,2} + |C_{1,2}|,\ B_{1,2-} = B_{1,2} - |C_{1,2}|, \\ D_{12+} &= |D_{12}| - |\bar{D}_{12}|,\ D_{12-} = |D_{12}| + |\bar{D}_{12}|, \end{aligned}$$

provided that the following phase conditions are fulfilled

$$\arg(C_1) - \arg(D_{12}) + \arg(\bar{D}_{12}) = 0,\ \arg(C_2) - \arg(D_{12}) - \arg(\bar{D}_{12}) = 0. \qquad (8)$$

Transformed matrix

$$\hat{A} = \begin{pmatrix} -(B_1 + |C_1|) & 0 & 0 & |D_{12}| - |\bar{D}_{12}| \\ 0 & -(B_1 - |C_1|) & |D_{12}| + |\bar{D}_{12}| & 0 \\ 0 & |D_{12}| + |\bar{D}_{12}| & -(B_2 - |C_2|) & 0 \\ |D_{12}| - |\bar{D}_{12}| & 0 & 0 & -(B_2 + |C_2|) \end{pmatrix}. \quad (9)$$

As special cases we have well-known results for particular processes when $C_1 = C_2 = \bar{D}_{12} = 0$ (parametric generation and amplification), $B_2 = C_2 = D_{12} = \bar{D}_{12} = 0$ (sub-harmonic generation), etc. In these processes it is always $B_{1,2} \geq 0$ as representing noise in single modes, whereas the quantum properties are characterized by $K = B_1 B_2 - |D_{12}|^2 < 0$ or $K = B_1^2 - |C_1|^2 < 0$, for classical cases $K \geq 0$. In our more general case mixing the basic processes, also $B_{1,2-}$ can be negative, which provides more general situations.

Generating function

The s-ordered generating function is obtained from the s-ordered characteristic function as

$$G_s(\lambda_1, \lambda_2, t) = \frac{1}{\pi^2 \lambda_1 \lambda_2} \int \int \exp\left(-\frac{|\beta_1|^2}{\lambda_1} - \frac{|\beta_2|^2}{\lambda_2} \right) C_s(\beta_1, \beta_2, t) \mathrm{d}^2\beta_1 \mathrm{d}^2\beta_2, \tag{10}$$

where λ_1 and λ_2 are parameters of the generating function and the s-ordered characteristic function is defined by substituting $B_{1,2s\pm} = B_{1,2\pm} + (1-s)/2$ instead of $B_{12\pm}$.

For M equally behaved modes (temporal, spatial and polarization in the spirit of Mandel-Rice formula)

$$\begin{aligned} G_s(\lambda_1,\lambda_2,t) &= G_{s+}(\lambda_1,\lambda_2,t)G_{s-}(\lambda_1,\lambda_2,t) \\ &= (1+\lambda_1 B_{1s+}+\lambda_2 B_{2s+}+\lambda_1\lambda_2 K_{s+})^{-M/2} \\ &\quad \times(1+\lambda_1 B_{1s-}+\lambda_2 B_{2s-}+\lambda_1\lambda_2 K_{s-})^{-M/2}, \end{aligned} \tag{11}$$

i.e. we have the product of two generating functions for two irreducible processes.

Joint photon-number distribution

Joint photon-number distribution $p(n_1, n_2, t)$ for multi-Gaussian field with M degrees of freedom can be derived in the form:

$$p(n_1, n_2, t) = \sum_{k=0}^{n_1} \sum_{l=0}^{n_2} p_+(k, l, t) p_-(n_1 - k, n_2 - l, t), \tag{12}$$

where

$$\begin{aligned} p_\pm(n_1, n_2, t) &= \frac{(-1)^{n_1+n_2}}{n_1! n_2!} \frac{\partial^{n_1+n_2} G_{\mathcal{N}\pm}(\lambda_1, \lambda_2, t)}{\partial \lambda_1^{n_1} \partial \lambda_2^{n_2}} \bigg|_{\lambda_1=\lambda_2=1} \\ &= \frac{1}{\Gamma(M/2)} \frac{(B_{1\pm} + K_\pm)^{n_1} (B_{2\pm} + K_\pm)^{n_2}}{(1 + B_{1\pm} + B_{2\pm} + K_\pm)^{n_1+n_2+M/2}} \\ &\times \sum_{r=0}^{\min(n_1,n_2)} \frac{\Gamma(n_1 + n_2 + M/2 - r)}{r!(n_1 - r)!(n_2 - r)!} \\ &\times \frac{(-K_\pm)^r [(1 + B_{1\pm} + B_{2\pm} + K_\pm)]^r}{[(B_{1\pm} + K_\pm)(B_{2\pm} + K_\pm)]^r}. \end{aligned} \tag{13}$$

Determinants $K_{\pm} = B_{1\pm}B_{2\pm} - |D_{12\pm}|^2$ are crucial for the judgement of classicality or nonclassicality of the field. Negative values of the determinants $K_{\pm}$ mean that a given field cannot be described classically, which is the case of the field under discussion composed of photon pairs. In (13), the quantities $B_{1,2\pm} + K_{\pm}$ can be considered as characteristics of fictitious noise present in the fields, giving declination from the purity of the process, i.e. from the diagonal form of $p(n_1, n_2, t)$ when photons are ideally paired; physically they are positive. However, in this case they characterize virtual distributions giving the actual distribution after the convolution process, which means that they may take on also negative values in some cases. Even if the corresponding partial joint photon-number distribution $p_{-}(n_1, n_2, t)$ can be unphysical taking on negative values, the resulting convoluted distribution is actual quantum oscillating physical distribution (this is an analogue of one-dimensional case), i.e. this leads to quantum oscillations in the photon-number distribution $p(n_1, n_2, t)$ (see Figs. 2–4).

Joint wave quasidistributions

The s-ordered joint wave quasidistribution $P_s(W_1, W_2, t)$ is obtained from the partial quasidistributions in the form of convolution

$$P_s(W_1, W_2, t) = \int_0^{W_1} \int_0^{W_2} P_{s+}(W_1', W_2', t) P_{s-}(W_1 - W_1', W_2 - W_2', t) \mathrm{d}W_1' \mathrm{d}W_2'. \tag{14}$$

Provided that the s-ordered determinant $K_{s\pm} = B_{1s\pm}B_{2s\pm} - |D_{12\pm}|^2 = K_\pm + (1-s)(B_{1s\pm} + B_{2s\pm})/2 + (1-s)^2/4$ is positive the s-ordered joint quasidistribution $P_{s\pm}(W_1, W_2)$ of integrated intensities exists as an ordinary function which cannot take on negative values:

$$\begin{aligned} P_{s\pm}(W_1,W_2) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} \exp(-\mathrm{i}s_1W_1 - \mathrm{i}s_2W_2)G_{s\pm}(-\mathrm{i}s_1, -\mathrm{i}s_2)\mathrm{d}s_1\mathrm{d}s_2 \\ &= \frac{1}{\Gamma(M/2)K_{s\pm}^{M/2}}\left(\frac{K_{s\pm}^2 W_1W_2}{|D_{12\pm}|^2}\right)^{(M/2-1)/2} \\ &\quad \times \exp\left[-\frac{(B_{2s\pm}W_1/B_{1s\pm} + W_2)B_{1s\pm}}{K_{s\pm}}\right] \\ &\quad \times I_{M/2-1}\left(2\sqrt{\frac{|D_{12\pm}|^2W_1W_2}{K_{s\pm}^2}}\right). \end{aligned} \tag{15}$$

This is some squeezed-form Gaussian distribution.

If the s-ordered determinant $K_{s\pm}$ is negative, the joint signal-idler quasidistribution $P_{s\pm}$ of integrated intensities exists in general as a generalized function that can take on negative values or can even have singularities:

$$P_{s\pm}(W_1, W_2, t) = \frac{(W_1 W_2)^{(M/2-1)/2}}{\pi\Gamma(M/2)(-K_{s\pm})^{1/2}(B_{1s\pm}B_{2s\pm})^{M/4}} \times \exp\left(-\frac{W_1}{2B_{1s\pm}} - \frac{W_2}{2B_{2s\pm}}\right) \mathrm{sinc}\left[A\left(\frac{B_{2s\pm}}{B_{1s\pm}}W_1 - W_2\right)\right] \quad (16)$$

where $\mathrm{sinc}(x) = \sin(x)/x$, $A = (-K_{s\pm}B_{2s\pm}/B_{1s\pm})^{-1/2}$.
Oscillating behavior is typical for the quasidistribution $P_{s\pm}$ written in (16). The formula (16) is exact because we can show that the poles in the upper half complex plane give generally no contribution to the corresponding integral (see next section).

There exist threshold values $s_{\mathrm{th}\pm}$ of the ordering parameter s for given values of parameters $B_{1,2\pm}$ and $D_{12\pm}$ determined by $K_{s\pm} = 0$ (the corresponding joint s-ordered wave distribution is diagonal):

$$s_{\mathrm{th}\pm} = 1 + B_{1\pm} + B_{2\pm} - \sqrt{(B_{1\pm} + B_{2\pm})^2 - 4K_{\pm}}; \qquad (17)$$

$-1 \leq s_{\mathrm{th}\pm} \leq 1$. Quasidistributions $P_{s\pm}$ for $s \leq s_{\mathrm{th}\pm}$ are ordinary functions with non-negative values whereas those for $s > s_{\mathrm{th}\pm}$ are generalized functions with negative values and oscillations.

This expression can also be written in the form $1 + B_{1\pm} + B_{2\pm} - \sqrt{(B_{1\pm} - B_{2\pm})^2 + 4D_{12\pm}^2}$ directly related to the principal squeeze parameters for equally behaved modes $\lambda_{\pm} = 1 + 2(B_{\pm} - |D_{12\pm}|)$.

In this sense the quasidistribution behaves in a quantum way for s between the threshold value $s_{\rm th}$ and 1 (Glauber-Sudarshan quasidistribution), for s equal or less than the threshold value, the vacuum fluctuations related to the value s compensate nonclassicality in the field (expressed e.g. by the squeezing of vacuum fluctuations in the field) and the quasidistribution behaves classically.
Considering the basic process only, even if negative probabilities, which can be reconstructed from experimental data, represent only qualitative phenomenon reflecting debt of probabilities in richer qunatum dynamics compared to classical dynamics for us and we do not interpret them directly, they have direct consequences in the discrete region in $p(n_1, n_2)$.

In the quantum region $K < 0$ and $K + B \geq 0$, i.e. $K + B < B$; in the pure case we have $K = -B$ and $K + B = 0$, which leads to the diagonal Mandel-Rice formula for $p(n_1, n_2)$ giving the oscillating sum-number distribution with zero odd values; in general we have quantum oscillations and squeezed-form distribution up to the border between quantum and classical regions for $K = 0$ ($K + B = B$), which reflects negative probabilities.
In the classical region $K \geq 0$, i.e. $K + B \geq B$, quasidistributions behave classically and forms of $p(n_1, n_2)$ change among that for $K = 0$ and the isotropic case $p(n_1, n_2) = p(n_1)p(n_2)$.

The other quantities characterizing nonclassical behavior:

- quantum entanglement $K_+K_{\mathcal{A}+} + K_-K_{\mathcal{A}-} < 0$ ($K_{\mathcal{A}\pm}$ are the corresponding determinants for antinormal operator ordering, i.e. $B_\pm$ is substituted by $B_\pm + 1$).
- conditional number distributions $p_{c,2}(n_2; n_1)$ which are sub-Poissonian giving the Fano factor $F_{c,2} < 1$.
- sub-Poissonian difference-number distribution $p_-(n)$.
- sum- and difference-wave quasidistributions $P_{s\pm}(W)$ exhibiting classical and quantum behavior, respectively.
- the principal squeezing $\lambda = 1 + B_1 + B_2 - 2\mathrm{Re}\bar{D}_{12} - |C_1 + C_2 + 2D_{12}| < 1$ (for the same modes $\lambda \geq s_{\mathrm{th}-}$).
- sub-shot-noise behavior $R = 1 + (K_+ + K_-)/2B < 1$ (the same modes).

Forms of quasidistributions

When we simplify our denotation of the quantities $B_{1,2s\pm}$, $D_{12\pm}$, $K_{s\pm}$ to $B_{1,2s}$, $D_{1,2}$, K_s, the above solutions are two of eight possibilites to have $B_{1,2s}$ and K_s positive or negative (we consider the most important first two cases):

1. $B_{1,2s} > 0$, $K_s > 0$

In this case the integrals

$$P_s(W_1, W_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \frac{\exp(-\mathrm{i}s_1 W_1 - \mathrm{i}s_2 W_2)}{[1 - \mathrm{i}s_1 B_{1s} - \mathrm{i}s_2 B_{2s} - s_1 s_2 K_s]^M} \mathrm{d}s_1 \mathrm{d}s_2 \qquad (18)$$

with respect to variables s_1 and s_1 are in the lower complex half-plane $\Pi^{(-)}$ and applying the Cauchy integral two times, we obtain the above I_M-distribution (15) which is regular and non-negative (squeezed-form Gaussian distribution).

We see this writing the polynomial in the denominator as

$$1 - \mathrm{i}s_1 B_{1s} - \mathrm{i}s_2 B_{2s} - s_1 s_2 K_s = -\left(\mathrm{i}B_{1s} + s_2 K_s\right) \times \left[s_1 + \frac{s_2 |D_{12}|^2 + \mathrm{i}(B_{1s} + s_2^2 B_{2s} K_s)}{B_{1s}^2 + s_2^2 K_s^2} \right]. \qquad (19)$$

The pole is in $\Pi^{(-)}$ and under these conditions the system behaves classically.

2. $B_{1,2s} > 0$, $K_s < 0$
This is the standard quantum case. To have the pole in $\Pi^{(-)}$ in the integration along s_1 to use the Cauchy integral, it must be

$$B_{1s} - s_2^2 B_{2s}(-K_s) > 0, \tag{20}$$

i.e. s_2 is filtered out of the interval $(-A, +A)$, $A = (B_{1s}/B_{2s}(-K_s))^{1/2}$ and we arrive at the integral

$$P_s(W_1, W_2) = \frac{W_1^{M-1}}{2\pi\Gamma(M)} \int_{-A}^{+A} \frac{\exp[-\mathrm{i}s_2 W_2 - \frac{1-\mathrm{i}s_2 B_{2s}}{-\mathrm{i}s_2 K_s + B_{1s}} W_1]}{(-\mathrm{i}s_2 K_s + B_{1s})^M} \mathrm{d}s_2. \tag{21}$$

Now we can show that the pole $s_2 = -\mathrm{i}B_{1s}/K_s$ lying in the upper half-plane $\Pi^{(+)}$ in this case gives no contribution to the integral (21) which is analytic in the lower half-plane $\Pi^{(-)}$, the integral is zero excepting $W_2 = 0$ where it is singular (the norm is unity). If we perform the integral (21) with respect to W_2 (we have a generalized function) and use one factor from the denominator, we have

$$\frac{1}{-\mathrm{i}s_2(-\mathrm{i}s_2K_s + B_{1s})} = \left(\frac{1}{-\mathrm{i}s_2} - \frac{K_s}{-\mathrm{i}s_2K_s + B_{1s}}\right)\frac{1}{B_{1s}}. \tag{22}$$

The integral involving the second term is zero because taking it from $-\infty$ to $+\infty$ (the integral for the filtered values of s_2 is zero because this means the opposite inequality in (20) and the pole lies in the upper half-plane), the pole is at $s_2 = -\mathrm{i}B_{1s}/K_s$ in $\Pi^{(+)}$ and performing the derivative with respect to W_2 returning back, we have the same integral with the index of denominator decreased by one; hence one factor $-\mathrm{i}s_2K_s + B_{1s}$ in the denominator is replaced by B_{1s}.

Successively we replace all these factors by B_{1s} including the corresponding terms in the exponential function decomposing it into the series. We finally obtain exactly the above sinc-quasidistribution (16) after explicit symmetrization (we change the order of integrations along s_1 and s_2 obtaining the same result, multiply the results and take the square root), i.e.

$$P_s(W_1, W_2) = \frac{(W_1 W_2)^{(M-1)/2}}{\pi\Gamma(M)(-K_s)^{1/2}(B_{1s}B_{2s})^{M/2}} \exp\left(-\frac{W_1}{2B_{1s}} - \frac{W_2}{2B_{2s}}\right) \times \mathrm{sinc}\left[A(\frac{B_{2s}}{B_{1s}}W_1 - W_2)\right]. \tag{23}$$

(We have started by the integration along s_1. If we start with s_2, everything is valid changing mutually indices 1 and 2.)

One-dimensional process

As an illustration we can now give the simplest case of the single mode second subharmonic generation ($\lambda_1 = \lambda$, $\lambda_2 = 0$, $B_1 = B$, $C_1 = C$, $B_2 = C_2 = D_{12} = \bar{D}_{12} = 0$).
(i) Spontaneous process
In this case the normal generating function has the form

$$G_{\mathcal{N}}(-\mathrm{i}s) = \frac{1}{[1 - \mathrm{i}s(B + |C|)]^{M/2}[1 - \mathrm{i}s(B - |C|)]^{M/2}} \tag{24}$$

for the degenerate case (for M even when the cross-correlation coefficient D_{12} stands instead of C we also have the generating function for non-degenerate case). Clearly the second-order polynomial in the denominator can be written as $1 - 2\mathrm{i}Bs - Ks^2$ (s is now Fourier variable and not s-ordering parameter; generalization to the s-ordering is straightforward writing $B_s = B + (1 - s)/2$ instead of B), where the determinant $K = B^2 - |C|^2$ (in non-degenerate case $K = B_1B_2 - |D_{12}|^2$).

If the field behaves nonclassically ($K < 0$), the mean number of quantum-noise photons $F = B + |C|$ is positive and the corresponding distribution is the Rayleigh gamma distribution, whereas $E = B - |C|$ is negative, leading to sub-Poisson behavior, squeezing of vacuum fluctuations and quantum oscillations in photon-number distribution. 35 years ago we were not successful to obtain the quasidistribution in the framework of one dimension. Generalizing the polynomial to the two dimensional form as $1 - \mathrm{i}Bs_1 - \mathrm{i}Bs_2 - s_1 s_2 K$ and performing the inverse Fourier transforms after s_1 and s_2 determining the Glauber-Sudarshan wave distribution $P_{\mathcal{N}}(W_1, W_2)$, we obtain again filtering of values of s_2 for $K < 0$ to have the Cauchy integral after s_1 with the pole in $\Pi^{(-)}$ as above, giving the frequency filtering $|s_2| \leq (-K)^{-1/2}$. Thus we have the partial integral

$$P_{\mathcal{N}-}(W) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp(-\mathrm{i}sW)}{(1 - \mathrm{i}sE)^{M/2}} \mathrm{d}s. \qquad (25)$$

Using the well-know identity ($P\frac{1}{s}$ means the principal value of the integral) $\int_0^\infty \exp(-\mathrm{i}sW)\mathrm{d}W = -\mathrm{i}P\frac{1}{s} + \pi\delta(s)$, we verify that $\int_0^\infty P_{\mathcal{N}-}(W)\mathrm{d}W = 1$. Integrating and deriving this integral with respect to W as above and using the decomposition

$$\frac{1}{-\mathrm{i}s\,(1-\mathrm{i}sE)} = \frac{1}{-\mathrm{i}s} - \frac{E}{1-\mathrm{i}sE}, \tag{26}$$

we obtain successively after integration in $\Pi^{(-)}$ and taking only not filtered values of s:

$$P_{\mathcal{N}-}(W) = \frac{\sin(AW)}{\pi W}, \; A = (-K)^{-1/2}. \tag{27}$$

The resulting quasidistribution is the convolution of the Rayleigh gamma distribution related to F,

$$P_{\mathcal{N}+}(W) = \frac{W^{M/2-1}}{\Gamma(M/2)F^{M/2}} \exp\left(-\frac{W}{F}\right), \tag{28}$$

and of this sinc-quasidistribution, taking on negative values. For non-degenerate process $E = B_2 - |D_{12}|$, $F = B_1 + |D_{12}|$, $A = (-B_1/B_2K)^{1/2}$.

(ii) Stimulated process
In this case

$$G_{\mathcal{N}-}(-\mathrm{i}s) = \frac{1}{(1-\mathrm{i}sE)^{M/2}} \exp\left(\frac{\mathrm{i}s\mathcal{A}}{1-\mathrm{i}sE}\right), \tag{29}$$

where E is the above quantum noise component and $\mathcal{A}$ is a coherent component related to the initial field. Decomposing the exponential function and applying the above arguments, we arrive at the shifted sinc-quasidistribution

$$P_{\mathcal{N}-}(W) = \frac{\sin(A(W-\mathcal{A}))}{\pi(W-\mathcal{A})}. \tag{30}$$

The resulting quasidistribution is the convolution of the regular I_{M-1}-distribution related to F,

$$P_{\mathcal{N}+}(W) = \frac{1}{F}\left(\frac{W}{\mathcal{B}}\right)^{(M/2-1)/2} \exp\left(-\frac{W+\mathcal{B}}{F}\right) I_{M/2-1}\left(2\frac{\sqrt{W\mathcal{B}}}{F}\right), \quad (31)$$

and of this sinc-quasidistribution, taking on negative values; here $\mathcal{B}$ is another partial coherent component related to the initial field.

Illustrations

We have adopted experimental data obtained in the Joint Laboratory of Optics in Olomouc. The wave quasidistribution $P_s(W_1, W_2)$ is shown in Fig. 1.

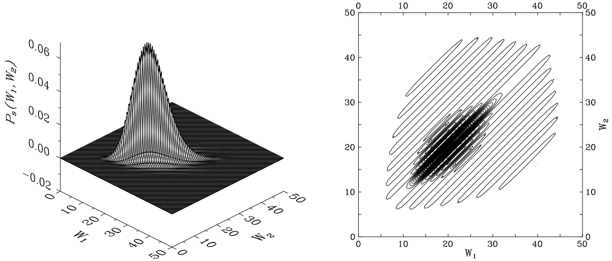

Fig. 1: Joint resulting wave quasidistribution $P_s(W_1, W_2)$ (left) and its contour curves (right), $s = 0.6$. In the classical region $K_{\pm} > 0$ a non-negative squeezed-form Gaussian distribution is obtained.

Solving the system of equations of motion, we arrived for the "optimalized" value for nonclassical behavior $g_1 t = 0.25$ choosing $g_3/g_1 = g_4/g_1 = g_2/g_1 = 0.1$, relative damping constants $\gamma_1/g_1 = \gamma_2/g_1 = 0.1$, and external reservoir noise $\langle n_{\mathrm{d1}} \rangle = \langle n_{\mathrm{d2}} \rangle = 1$. It was $K_\pm < 0, K_+ + B_+ > 0$, but $K_- + B_- < 0$, which virtually can occur leading to strong quantum oscillations and to negative values of the distribution $p_-(n_1, n_2)$ and to the resulting physical distribution $p(n_1, n_2)$ as the convolution of p_+ and p_- (both are nonclassical).

Fig. 2 provides quantum oscillating actual joint photon-number distribution obtained as the convolution of the partial virtual distributions p_+ and p_- given in Figs. 3 and 4, respectively; in Fig. 4 high negative oscillations of p_- are exhibited caused by negative values of $K_- + B_-$. In the pure case ($K_\pm = -B_\pm$) we have a line. This is analogous to the one-dimensional case demonstrated 35 years ago.
One can follow time evolution of the quasidistributions observing first a classical distribution at the beginning of the interaction developing as a regular distribution with $K_s > 0$ and with progressing squeezing of the form to one line with $K_s = 0$ (classical diagonal distribution) and then successively demonstrating quantum oscillations and negative values when $K_s < 0$ approaching finally some steady-state squeezed-form of quasidistribution with increasing period of oscillations.

This approach has been applied to the interpretation of experimental data from various experiments.

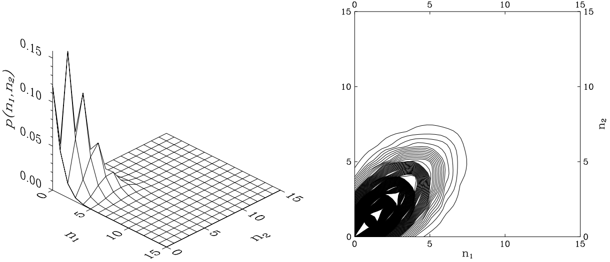

Fig. 2: Time evolution of joint photon-number distribution $p(n_1, n_2)$ (left) and its contour curves (right).

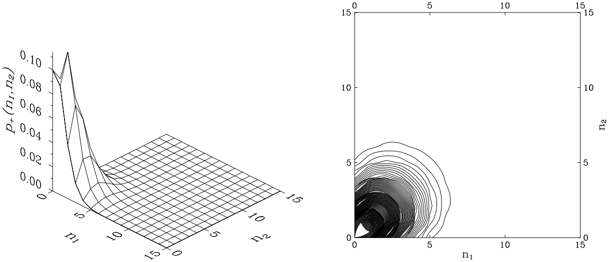

Fig. 3: Time evolution of partial joint photon-number distribution $p_+(n_1, n_2)$ (left) for quantities as in Fig. 2 and its contour curves (right).

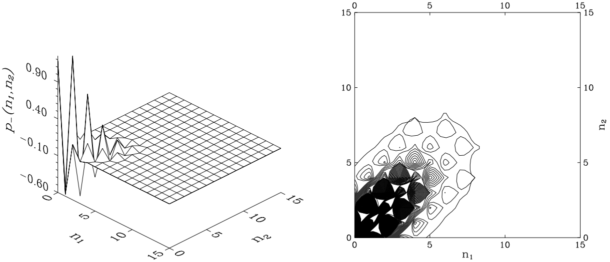

Fig. 4: Time evolution of partial joint photon-number distribution $p_-(n_1, n_2)$ (left) for quantities as in Fig. 2 and its cut curves (right).

Conclusion

- Demonstration of nonclassical behavior of wave quasidistributions for general optical parametric processes.
- Illustration of the system behavior by means of joint photon-number and wave quasidistributions.
- The important role of the sinc-quasidistributions in this description.
- All above results can be generalized to include stimulating fields (coherent, chaotic, squeezed).

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